Topic — Problem based on linear equation of 2nd order with variable co-efficients By Method of variation of parameters

Class — B.Sc. III (Maths HONS)

Date

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(1) Solve the Equation Applying the Method of variation

$$\frac{d^2y}{dx^2} + y = a\sec x$$

Sol^n $\longrightarrow$ Given Differential Equation is

$$\frac{d^2y}{dx^2} + y = a\sec x$$

The Auxiliary Equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1 = i^2$$

$$\Rightarrow m = \pm i$$

$\therefore$ C.F. is $= A\cos x + B\sin x$

Here A and B are functions of x. Now we have to Determine the values of A and B

these are given by

$$A_1 u + B_1 v = 0$$

and

$$A_1 u_1 + B_1 v_1 = R$$

$u = \cos x \quad v = \sin x \quad R = a\sec x$

$$A_1 \cos x + B_1 \sin x = 0 \quad \text{——— (I)}$$

$$-A_1 \sin x + B_1 \cos x = a\sec x$$

$$\Rightarrow -A_1 \sin x + B_1 \cos x = \frac{1}{\sin x} \quad \text{——— (II)}$$

On solving $A_1 = -1 \qquad B_1 = \cot x$

$$\frac{dA}{dx} = -1 \qquad \frac{dB}{dx} = \cot x$$

Putting these values ...

$y = A\cos x + B\sin x$

$= (-x + c_1)\cos x + (\log\sin x + c_2)\sin x$

$= c_1\cos x + c_2\sin x - x\cos x + \sin x \log\sin x$

(2) Solve. $\frac{d^2y}{dx^2} + y = x$

Given Equation

$$\frac{d^2y}{dx^2} + y = x$$

The Auxiliary Equation is

$D^2 + 1 = 0$

$D^2 = -1 = i^2$

$D = \pm i$

$\therefore$ CF is $y = A\cos x + B\sin x$

where A and B are functions of x

Here $u = \cos x$ $\quad v = \sin x$

$u_1 = -\sin x$ $\quad v_1 = \cos x$

This A and B satisfy the equations

$A_1 u + B_1 v = 0$ and $A_1 u_1 + B_1 v_1 = R$

$\Rightarrow A_1\cos x + B_1\sin x = 0$ and $-A_1\sin x + B_1\cos x = x$

Solving this

$A_1 = -x\sin x$ $\quad B_1 = x\cos x$

$\frac{dA}{dx} = -x\sin x$ $\quad \frac{dB}{dx} = x\cos x$

$\int dA = -\int x\sin x\,dx$ $\quad \int dB = \int x\cos x\,dx$

$$A = -\left[x\int \sin x\,dx - \int\left(\int \sin x\,dx \cdot \frac{dx}{dx}\right)dx\right]$$

$$= -\left[x(-\cos x) + \int \cos x\,dx\right]$$

$$= -\left[-x\cos x + \sin x\right] + C_1$$

$$= x\cos x - \sin x + C_1$$

$$B = \int x\cos x\,dx$$

$$= x\int \cos x\,dx - \int\left(\int \cos x\,dx \cdot \frac{dx}{dx}\right)dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C_2$$

This Required Solution is

$$y = A\cos x + B\sin x$$

$$= (x\cos x - \sin x + C_1)\cos x + (x\sin x + \cos x + C_2)\sin x$$

$$= x\cos^2 x - \sin x\cos x + C_1\cos x + x\sin^2 x + \sin x\cos x + C_2\sin x$$

$$= x + C_1\cos x + C_2\sin x$$

Ex. 3. Solve by method of variation of Parameters the Equation

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Solution – The given Differential Equation is.

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

The Auxiliary Equation is

$$m^2-1=0$$

$$\Rightarrow m^2=1$$

$$\Rightarrow m=\pm 1$$

this C.F is $y = Ae^{x} + Be^{-x}$ —— (I)

Where A and B are functions of x which we have to determine

these are given by

$$A_1 u + B_1 v = 0$$

$$A_1 u_1 + B_1 v_1 = R$$

Here $u = e^{x}$, $u_1 = e^{x}$

$v = e^{-x}$, $v_1 = -e^{-x}$

$R = \frac{2}{1+e^{x}}$

i.e $A_1 e^{x} + B_1 e^{-x} = 0$ —— (II)

$A_1 e^{x} - B_1 e^{-x} = \frac{2}{1+e^{x}}$ —— (III)

On solving (II) and (III)

$$A_1 = \frac{1}{e^{x}(1+e^{x})} \qquad B_1 = -\frac{1}{e^{-x}(1+e^{x})}$$

$$= -\frac{e^{x}}{1+e^{x}}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{e^{x}(1+e^{x})} \qquad \frac{dB}{dx} = -\frac{e^{x}}{1+e^{x}}$$

put $e^{x} = t$

$e^{x}dx = dt$

$dx = \frac{dt}{e^{x}} = \frac{dt}{t}$

$$\int dA = \int \frac{1}{e^{x}(1+e^{x})}\,dx$$

$$A = \int \frac{dt}{t\cdot t(1+t)}$$

$$= \int \frac{dt}{t^2(1+t)}$$

$$= \int \left(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{t+1}\right) dt$$

$$= -\frac{1}{t} - \log t + \log(t+1) + c_1$$

$$= -\frac{1}{e^x} - \log_e e^x + \log(e^x+1) + c_1$$

$$= -e^{-x} - x + \log(1+e^x) + c_1$$

Also $\dfrac{dB}{dx} = -\dfrac{e^x}{1+e^x}$

$$\int dB = -\int \frac{e^x}{1+e^x}\, dx$$

$$B = -\log(1+e^x) + C_2$$

∴ Complete Solution

$$y = \left[-e^{-x} - x + \log(1+e^x) + c_1\right] e^{x}$$
$$+ \left[-\log(1+e^x) + C_2\right] e^{-x}$$

$$= -1 - xe^x + e^x \log(1+e^x) + c_1 e^x$$
$$- e^{-x} \log(1+e^x) + C_2 e^{-x}$$